Some notations and properties for parametric quantum circuits on superconducting qubits

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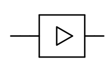
Native Gates

Rigetti, IBM and Google quantum computers are all based on superconducting qubits and have similar native gate sets. We will consider here the set of universal quantum gates CZ, RZ and SQRTX –that are native gates for the three hardware systems. Needless to say, another native operation considered here is the measurement (along the Z-axis).

Square root of Pauli X gate

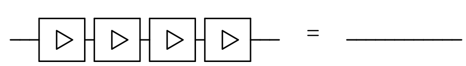
This gate is a $\pi/2$ rotation about the Pauli X-axis and is noted $SQRTX$, $X^{\frac{1}{2}}$, $X(\frac{\pi}{2})$, $R_x(\frac{\pi}{2})$, $\sqrt{X}$, or $\sqrt{NOT}$.
To avoid visual cluttering, in the graphical representation, we will use:



Matrix operator:

$$\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

From the matrix we can see that $X^{\frac{1}{2}}.X^{\frac{1}{2}}.X^{\frac{1}{2}}.X^{\frac{1}{2}} = -I$, where I is the identity operator. However, when measuring, the minus sign does not affect the outcome.
Hence, applying $\sqrt{NOT}$ consecutively 4 times is equivalent to doing nothing:

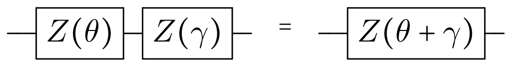


Rotations about the Pauli Z-axis

This parametric gate is noted $R_z(\theta)$ or $Z(\theta)$ with matrix:

$$\begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix}$$

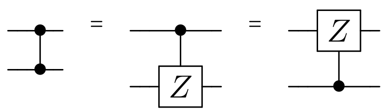
As expected two consecutive rotations about the same axis can be written as a single rotation:



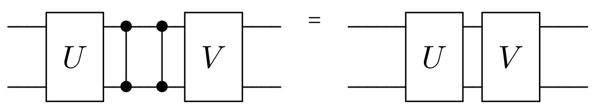
It is interesting to note that $Z(2\pi) = -I$. But, as noted before, the minus sign does not affect the measurement. So, for all practical purposes, Z rotation has a 2π period.

Controlled-Z Gate

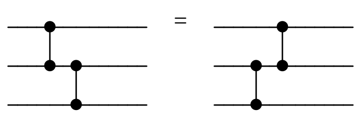
The CZ gate (also called $CPhase$) is a two-qubit operation (i.e. an entangling gate). CZ is symmetric regarding the qubits, hence the following graphical notation:



It is also interesting to note that CZ is an involution:

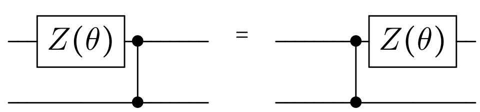


Also the controlled-Z gate is commutative:

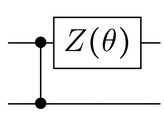


Combining Z-rotation with other gates

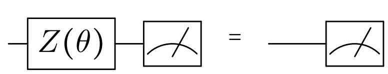
Commutativity with CZ



This means $Z(\theta)$ influence do not propagate on the other qubit so for clarity we will always use:



Measurement



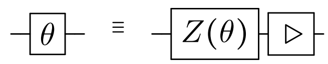
Doing a Z-rotation just before a measurement is useless...

Introducing the *input* gate

From the preceding statements it appears that in meaningful circuits a Z-rotation is always followed by a $\sqrt{X}$ gate. We can then introduce a parametric gate, the input gate, noted $IN(\theta)$ and defined as:

$$IN(\theta) = \sqrt{X}.Z(\theta)$$

The parameter θ is called the input value. The graphical representation of $IN(\theta)$ is simple and compact:



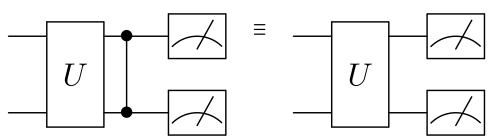
With the introduction of the input gate, the $Z(\theta)$ gate is not useful anymore. Also, as $\sqrt{X} = IN(0)$, we can consider the $\sqrt{X}$ gate is a particular input gate.

Input gate matrix:

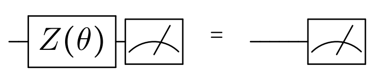
$$\frac{1}{2}\begin{pmatrix} (1+i)e^{-\frac{i\theta}{2}} & (1-i)e^{\frac{i\theta}{2}} \\ (1-i)e^{-\frac{i\theta}{2}} & (1+i)e^{\frac{i\theta}{2}} \end{pmatrix}$$

Gates before measurements

A controlled-Z entanglement just before measurements is useless:



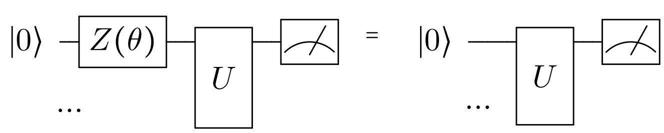
The same –as seen before– is true for Z-rotations:



Gates after initialization

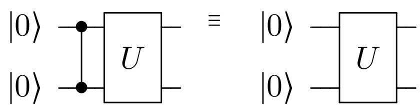
In actual hardware qubits are initialized at $|0\rangle$

Z-rotation as first gate

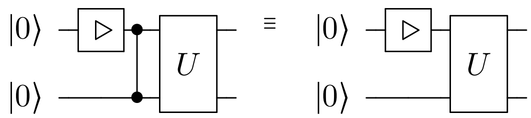


The effects of a Z-rotation on an initial qubit are not visible after measurement operations.

Controlled-Z after $|0\rangle$



Adding a $\sqrt{X}$ gate does not help:



As expected, when the control qubit is at $|0\rangle$ the controlled gate has no effect and in CZ both qubits are control qubits.

Conclusion and shorthand notation for initialization

As seen above, in all meaningful circuits the initialization at $|0\rangle$ needs to be followed by a $\sqrt{X}$ gate. We can hence introduce a notation for the initialization of qubits:

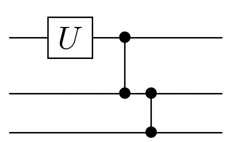
Or said differently, when using only CZ, $Z(\theta)$, $\sqrt{X}$ gates, qubits are initialized with:

$\frac{1}{2}\begin{pmatrix}1+i\\1-i\end{pmatrix}$

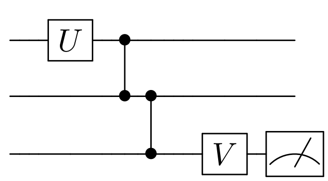
Combining controlled-Z gates

Information propagation through entanglement

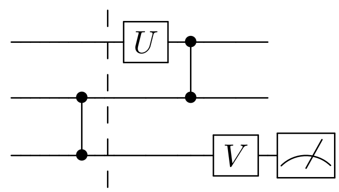
Question: does U operation has an effect on the 3$^{\text{rd}}$ qubit ?



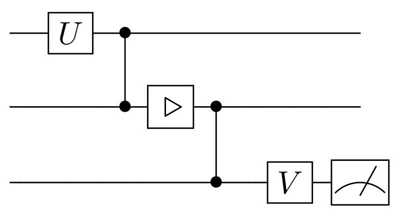
Or stated differently: Does U affect the measure on the 3$^{\text{rd}}$ qubit ? (after any V operation, to make sure the potential effect is not orthogonal to the measurement basis):



As CZ gates commute we can clearly see that two successive CZ gates cannot propagate information to the next qubit:

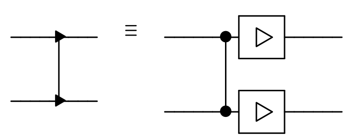


When adding $\sqrt{X}$ between the CZ gates, the information propagates and U does affect the measurement result:



Shorthand compact notation for common entanglement gate

From what precedes about CZ gates we can see it is useful to append $\sqrt{X}$ gates after a CZ gate and we can introduce this notation:

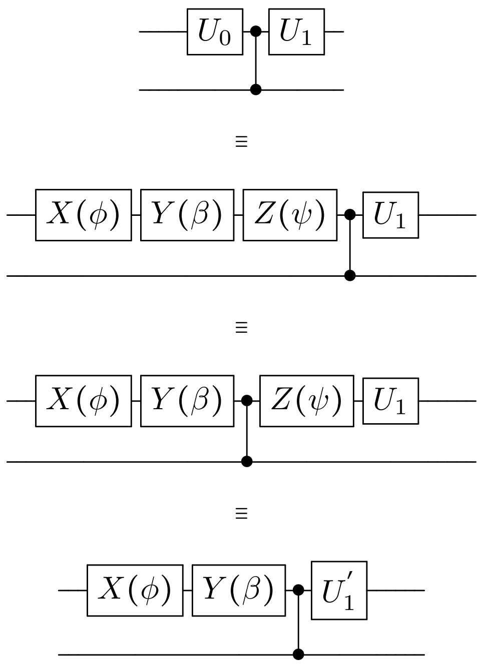


This gate can be also referred as the *common controlled* gate or CC gate, with matrix:

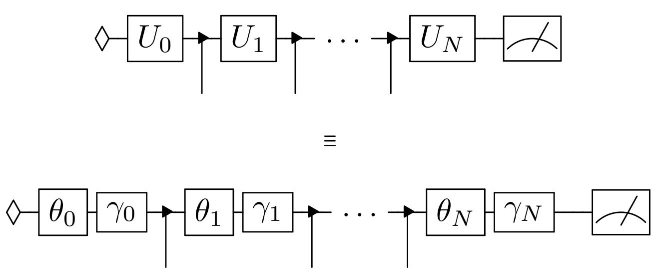
$$\frac{1}{2}\begin{pmatrix} i & 1 & 1 & i \\ 1 & i & -i & -1 \\ 1 & -i & i & -1 \\ -i & 1 & 1 & -i \end{pmatrix}$$

A qubit has two degrees of freedom

As we know any unitary U can be decomposed in 3 rotations $Z(\psi)Y(\beta)X(\phi)$, by induction we can see that rewrite any circuit with at most two rotations between CZ gates:



Or, using the gates defined here, the rewriting rule becomes:



For some sequence θ_i, γ_i of lenght N.

Summary of notations